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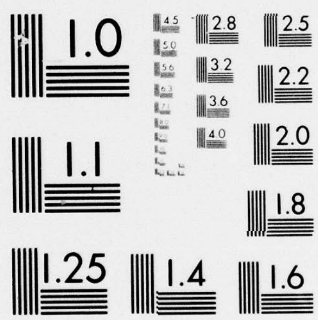

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THE INFORMATION MATRICES OF THE PARAMETERS OF MULTIPLE MIXED TIME SERIES*

by

H. Joseph Newton
State University of New York at Buffalo

TECHNICAL REPORT NO. 53

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THE INFORMATION MATRICES OF THE PARAMETERS OF

MULTIPLE MIXED TIME SERIES*

by

H. Joseph Newton

State University of New York at Buffalo
Amherst, New York

ABSTRACT

Closed form matrix equations are given for the information matrix of the parameters of the vector mixed autoregressive moving average time series model.

* Research supported in part by the Office of Naval Research.

1. Introduction

Consider the d-dimensional mixed autoregressive moving average process $\{\underset{\sim}{X}(t) , t \in Z\}$, Z the set of integers, of order (p,q) ,

$$\sum_{j=0}^{p} A(j)\, \underset{\sim}{X}(t-j) = \sum_{k=0}^{q} B(k)\, \underset{\sim}{\varepsilon}(t-k) , \quad t \in Z$$

where $A(0) \equiv B(0) \equiv I_d$, the d-dimensional identity matrix, and $\{\underset{\sim}{\varepsilon}(t) , t \in Z\}$ is a collection of uncorrelated, zero mean, d-dimensional random variables each having positive definite covariance matrix $\underset{\sim}{\Sigma}$.

The estimation of the $(d \times d)$ matrices $A(1),\ldots,A(p)$, $B(1),\ldots,B(q)$, and $\underset{\sim}{\Sigma}$ has received considerable attention in the recent literature (Akaike (1973a), Wilson (1973), Dunsmuir and Hannan (1976), for example). One of the most difficult computational problems involved in the estimation is that of determining the asymptotic covariance matrix V of the maximum likelihood estimators. Hannan (1970), p. 385 and p. 329 has given expressions for V for the case $p = 0$ or $q = 0$.

In this paper we derive closed form matrix expressions for V by using Whittle's (1953) formula for the Fisher information matrix of the parameters of a Gaussian multiple time series.

2. The Information Matrix of the Mixed Process

Let $\{\underset{\sim}{Y}(t)\ ,\ t \in Z\}$ be a zero mean Gaussian time series whose distribution depends on parameters $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_r)^T$, where A^T denotes the transpose of the matrix A . Let $f(\omega)$, $\omega \epsilon [-\pi,\pi]$ be the spectral density matrix of $\underset{\sim}{Y}(\cdot)$. Then given a sample realization $\underset{\sim}{y}(1),\ldots,\underset{\sim}{y}(T)$ of $\underset{\sim}{Y}(\cdot)$, Whittle (1953) shows that the maximum likelihood estimators $\hat{\underset{\sim}{\theta}}_T$ of $\underset{\sim}{\theta}$ are such that $\sqrt{T}\,(\hat{\underset{\sim}{\theta}}_T - \underset{\sim}{\theta})$ is asymptotically r-dimensional normal with mean zero and covariance matrix $V(\underset{\sim}{\theta}) = I^{-1}(\underset{\sim}{\theta})$ where the (j,k)th element of $I(\underset{\sim}{\theta})$ is given by

$$I_{jk}(\underset{\sim}{\theta}) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathrm{tr}\left[\frac{\partial f(\omega)}{\partial \theta_j}\, f^{-1}(\omega)\, \frac{\partial f(\omega)}{\partial \theta_k}\, f^{-1}(\omega)\right] d\omega\ , \tag{2.1}$$

where the notation $\frac{\partial A}{\partial b}$, b a scalar and $A = (A_{jk})$ and (n x m) matrix, denotes the (n x m) matrix of derivatives $\left(\frac{\partial A_{jk}}{\partial b}\right)$, and tr B denotes the trace of the matrix B .

Let $D = (D_{jk})$ be an (n x n) matrix and $U = U(D)$, $V = V(D)$ be matrix functions of D . Then (Neudecker (1969), for example)

$$\frac{\partial D}{\partial D_{jk}} = E_{jk}\ ,\quad \frac{\partial D^T}{\partial D_{jk}} = E_{jk}^T = E_{kj}\ , \tag{2.2}$$

where E_{jk} is the zero matrix except the (j,k)th element which is one. Also, the chain rule holds;

$$\frac{\partial UV}{\partial D_{jk}} = \frac{\partial U}{\partial D_{jk}} V + U \frac{\partial V}{\partial D_{jk}}\ . \tag{2.3}$$

Thus from $D^{-1}D = I_n$, where I_n denotes the nth order identity matrix, we have

$$\frac{\partial D^{-1}}{\partial a} = -D^{-1} \frac{\partial D}{\partial a} D^{-1} \quad . \tag{2.4}$$

Note that the trace operation satisfies

$$\mathrm{tr}\,(A + B) = \mathrm{tr}\,(A) + \mathrm{tr}\,(B)$$

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$$

From (2.4) we can write (2.1) as

$$I_{jk} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathrm{tr}\left[\frac{\partial f^{-1}}{\partial \theta_j} f \frac{\partial f^{-1}}{\partial \theta_k} f\right] d\omega \tag{2.5}$$

$$= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \mathrm{tr}\left[\frac{\partial f^{-1}}{\partial \theta_j} \frac{\partial f}{\partial \theta_k}\right] d\omega \quad , \tag{2.6}$$

where we have now deleted the argument of $f(\cdot)$ for convenience.

The spectral density matrix $f(\cdot)$ of the mixed model can be written (see Hannan (1970), p. 67)

$$f(\omega) = \frac{1}{2\pi} G^{-1}(e^{i\omega}) H(e^{i\omega}) \Sigma H^{*}(e^{i\omega}) G^{-*}(e^{i\omega}) \ , \quad \omega \epsilon [-\pi, \pi] \tag{2.7}$$

where the complex matrix polynomials $G(\cdot)$ and $H(\cdot)$ are given by

$$G(z) = \sum_{j=0}^{p} A(j)\, z^{j}$$

$$H(z) = \sum_{k=0}^{q} B(k)\, z^{k} \quad ,$$

and A^* denotes the complex conjugate transpose of the matrix A . We assume that the zeros of $\det\left(G(z)\right)$ and $\det\left(H(z)\right)$ are outside the unit circle so that the elements of $G^{-1}(z)$ and $H^{-1}(z)$ can be written as power series in z .

From (2.7) we write

$$2\pi f = G^{-1} H \underset{\sim}{\Sigma} H^* G^{-*} \equiv G^{-1} Q G^{-*} \tag{2.8}$$

$$\frac{1}{2\pi} f^{-1} = G^* Q^{-1} G \quad , \tag{2.9}$$

where G doesn't involve the $B(\cdot)$, $Q = H \underset{\sim}{\Sigma} H^*$ is Hermitian and mathematically independent of the $A(\cdot)$, $Q = \underset{\sim}{\Sigma}$ if $q = 0$, and we have deleted the arguments of all functions.

Clearly we must order the elements of the $A(\cdot)$ and $B(\cdot)$ into a vector $\underset{\sim}{\theta}$. However, we first find the element of the information matrix corresponding to $A_{jk}(v)$, $A_{\ell m}(u)$ (denoted $I[A_{jk}(v), A_{\ell m}(u)]$) by (2.5), the element corresponding to $B_{jk}(v)$, $B_{\ell m}(u)$ (denoted $I[B_{jk}(v), B_{\ell m}(u)]$) by (2.1), and the element corresponding to $A_{jk}(v), B_{\ell m}(u)$ (denoted $I[A_{jk}(v), B_{\ell m}(u)$ by (2.6)). Then we consider various orderings of the elements of the $A(\cdot)$ and $B(\cdot)$ to find a convenient expression for $I(\underset{\sim}{\theta})$.

$\underline{I[A_{jk}(v)\ ,\ A_{\ell m}(u)]}$

From (2.2), (2.3), and (2.5) we obtain

$$\frac{1}{2\pi}\ \frac{\partial f^{-1}}{\partial A_{jk}(v)} = \frac{\partial}{\partial A_{jk}(v)}\ G^{*}Q^{-1}G$$

$$= \frac{\partial G^{*}Q^{-1}}{\partial A_{jk}(v)}\ G + G^{*}Q^{-1}\ \frac{\partial G}{\partial A_{jk}(v)}$$

$$= E_{kj}Q^{-1}G\, e^{-iv\omega} + G^{*}Q^{-1}E_{jk}e^{iv\omega} \quad .$$

Thus

$$\frac{\partial f^{-1}}{\partial A_{jk}(v)}\ f = E_{kj}G^{-*}e^{-iv\omega} + f^{-1}G^{-1}\,E_{jk}f\, e^{iv\omega} \quad ,$$

and

$$\frac{\partial f^{-1}}{\partial A_{jk}(v)}\ f\ \frac{\partial f^{-1}}{\partial A_{\ell m}(u)}\ f = E_{kj}G^{-*}E_{m\ell}G^{-*}e^{-i(u+v)\omega} + 2\pi\, E_{kj}Q^{-1}\,E_{\ell m}f\, e^{-i(v-u)\omega}$$

$$+ 2\pi\, G^{*}Q^{-1}\,E_{jk}f\,E_{m\ell}G^{-*}e^{-i(u-v)\omega}$$

$$+ f^{-1}\,G^{-1}\,E_{jk}G^{-1}\,E_{\ell m}f\, e^{i(u+v)\omega} \quad .$$

Denote by D^{rs} the matrix D replaced by zeros except for the r^{th} row which is replaced by the s^{th} row, _i.e._ $D^{rs} = E_{rs}D$. Then

for matrices $D = (D_{jk})$, $C = (C_{\ell m})$ we have

$$\mathrm{tr}[E_{rs} D E_{tu} C] = \mathrm{tr}[D^{rs} C^{tu}] = D_{st} C_{ur}$$

Thus

$$\mathrm{tr}\left[\frac{\partial f^{-1}}{\partial A_{jk}(v)} f \frac{\partial f^{-1}}{\partial A_{\ell m}(u)} f\right] = \mathrm{tr}[E_{kj} G^{-*} E_{m\ell} G^{-*}] \, e^{-i(u+v)\omega}$$

$$+ 2\pi \, \mathrm{tr}[E_{kj} Q^{-1} E_{\ell m} f] \, e^{-i(v-u)\omega}$$

$$+ 2\pi \, \mathrm{tr}[E_{m\ell} Q^{-1} E_{jk} f] \, e^{-i(u-v)\omega}$$

$$+ \mathrm{tr}[E_{\ell m} G^{-1} E_{jk} G^{-1}] \, e^{i(u+v)\omega}$$

$$= G^{-*}_{jm} G^{-*}_{\ell k} e^{-i(u+v)\omega} + 2\pi \, Q^{-1}_{j\ell} f_{mk} e^{-i(v-u)\omega}$$

$$+ 2\pi \, Q^{-1}_{\ell j} f_{km} e^{-i(u-v)\omega} + G^{-1}_{mj} G^{-1}_{k\ell} e^{i(u+v)\omega} \quad .$$

We argue that the first and last terms integrate to zero, as follows: because the roots of $\det\big(G(z)\big) = 0$ are assumed strictly outside the unit circle, G^{-1}_{jk} , the (j,k)th element of G^{-1} , can be written as a power series (which allows the interchange of summation and integration in (2.10) below)

$$G_{jk}^{-1}(z) = \sum_{\ell=0}^{\infty} C_{jk}(\ell)\, z^{\ell}$$

and thus

$$\int_{-\pi}^{\pi} G_{mj}^{-1}\, G_{k\ell}^{-1}\, e^{i(u+v)\omega}\, d\omega$$

$$= \int_{-\pi}^{\pi} \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} C_{mj}(t)\; C_{k\ell}(r)\; e^{i(u+v+t+r)\omega}\, d\omega \tag{2.10}$$

$$= \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} C_{mj}(t)\; C_{k\ell}(r) \int_{-\pi}^{\pi} e^{i(u+v+t+r)\omega}\, d\omega \quad ,$$

and the integral is always zero since $(u+v+t+r) > 0$.

Thus we are left with

$$I[A_{jk}(v), A_{\ell m}(u)] = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathrm{tr}[C + C^{*}]\, d\omega \quad ,$$

where

$$C = 2\pi\, E_{kj}\, Q^{-1}\, E_{\ell m}\, f\, e^{-i(v-u)\omega} \quad .$$

Since the integral is real, we have

$$\begin{aligned} I[A_{jk}(v), A_{\ell m}(u)] &= \frac{1}{4\pi} \int_{-\pi}^{\pi} 2\, \mathrm{tr}[C]\, d\omega \\ &= \int_{-\pi}^{\pi} Q_{j\ell}^{-1}\, f_{mk}\, e^{-i(v-u)\omega}\, d\omega \quad . \end{aligned} \tag{2.11}$$

<u>$I[B_{jk}(v)\ ,\ B_{\ell m}(u)]$</u>

From (2.8) we have

$$2\pi \frac{\partial f}{\partial B_{jk}(v)} = \frac{\partial}{\partial B_{jk}(v)} G^{-1}H \ddagger H^{*}G^{-*}$$

$$= \frac{\partial G^{-1}H}{\partial B_{jk}(v)} \ddagger H^{*}G^{-*} + G^{-1}H \frac{\partial \ddagger H^{*}G^{-*}}{\partial B_{jk}(v)}$$

$$= G^{-1} E_{jk} \ddagger H^{*}G^{-*} e^{iv\omega} + G^{-1}H \ddagger E_{kj} G^{-*} e^{iv\omega} \quad .$$

Then

$$\frac{\partial f}{\partial B_{jk}(v)} f^{-1} \frac{\partial f}{\partial B_{\ell m}(u)} f^{-1} = G^{-1}E_{jk} \ddagger H^{*}Q^{-1} E_{\ell m} \ddagger H^{*}Q^{-1} G e^{i(u+v)\omega}$$

$$+ G^{-1} E_{jk} \ddagger H^{*}Q^{-1}H \ddagger E_{m\ell} Q^{-1}G e^{-i(u-v)\omega}$$

$$+ G^{-1} H \ddagger E_{kj}Q^{-1} E_{\ell m} \ddagger H^{*}Q^{-1}G e^{-i(v-u)\omega}$$

$$+ G^{-1} H \ddagger E_{kj} Q^{-1} H \ddagger E_{m\ell} Q^{-1}G e^{-i(u+v)\omega} \quad .$$

The trace of this is given by

$$\mathrm{tr}\left[\frac{\partial f}{\partial B_{jk}(v)} f^{-1} \frac{\partial f}{\partial B_{\ell m}(u)} f^{-1}\right] = \mathrm{tr}[C + C^{*}] + \mathrm{tr}[D + D^{*}] \ ,$$

where

$$C = E_{\ell m} H^{-1} E_{jk} H^{-1} e^{i(u+v)\omega}$$

$$D = E_{m\ell} Q^{-1} E_{jk} \Sigma e^{-i(u-v)\omega}$$

The integral of the first term again vanishes, and we have

$$I[B_{jk}(v), B_{\ell m}(u)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q_{j\ell}^{-1} \Sigma_{mk} e^{-i(v-u)\omega} d\omega \quad . \tag{2.12}$$

$\underline{I[A_{jk}(v)\ ,\ B_{\ell m}(u)]}$

To find the off block diagonal elements of I we obtain, using arguments similar to the ones above, that

$$\mathrm{tr}\left[\frac{\partial f^{-1}}{\partial A_{jk}(v)} \quad \frac{\partial f}{\partial B_{\ell m}(u)}\right] = \mathrm{tr}[C + C^*] + \mathrm{tr}[D + D^*] \quad ,$$

where

$$C = E_{\ell m} H^{-1} E_{jk} G^{-1} e^{i(u+v)\omega}$$

$$D = E_{kj} Q^{-1} E_{\ell m} \Sigma H^* G^{-*} e^{-i(v-u)\omega} \quad .$$

Thus

$$I[A_{jk}(v), B_{\ell m}(u)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q_{j\ell}^{-1} [\Sigma H^* G^{-*}]_{mk}\ e^{-i(v-u)\omega} d\omega \ , \tag{2.13}$$

where the (m,k)th element of the product $\Sigma H^* G^{-*}$ is denoted $[\Sigma H^* G^{-*}]_{mk}$.

So the integrands in the expressions for $I[A_{jk}(v), A_{\ell m}(u)]$, $I[B_{jk}(v), B_{\ell m}(u)]$, and $I[A_{jk}(v), B_{\ell m}(u)]$ are all of the form $C^{-1}_{j\ell} D_{mk} e^{-i(v-u)\omega}$ for some matrices C and D , i.e., we have the transformation of indices denoted

$$(j,k,v) , (\ell,m,u) \rightarrow (j,\ell,-v) , (m,k,u) \quad . \tag{2.14}$$

This transformation allows us to find general expressions for the information matrix for various orderings of the elements of the $A(\cdot)$ and $B(\cdot)$ matrices.

Ordering of $A(\cdot), B(\cdot)$

Define the partitioned matrices

$$A_1 = \left(A(1) \vdots \cdots \vdots A(p) \right) \quad ,$$

$$A_2 = \left(A^T(1) \vdots \cdots \vdots A^T(p) \right) \quad ,$$

$$B_1 = \left(B(1) \vdots \cdots \vdots B(q) \right) \quad ,$$

$$B_2 = \left(B^T(1) \vdots \cdots \vdots B^T(q) \right) \quad .$$

Then let

$$\underset{\sim}{\alpha}_1 = \text{vec}\,(A_1)$$

$$\underset{\sim}{\alpha}_2 = \text{vec}\,(A_1^T)$$

$$\underset{\sim}{\alpha}_3 = \text{vec}\,(A_2)$$

$$\underset{\sim}{\beta}_1 = \text{vec}\,(B_1^T)$$

$$\underset{\sim}{\beta}_2 = \text{vec}\,(B_2^T)$$

$$\underset{\sim}{\beta}_3 = \text{vec}\,(B_1)$$

$$J_{nm}(\omega) = (J_{jk}(\omega)) = e^{i(j-k)\omega}\ , \quad j = 1,\ldots,n, k = 1,\ldots,m \quad ,$$

where vec (A) denotes the operation of stacking the columns of A , i.e., if $A = (\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_n)$, $\text{vec}\,(A) = \left(\underset{\sim}{a}_1^T \vdots \cdots \vdots\, \underset{\sim}{a}_n^T\right)^T$.

Define the Kronecker product of the $n \times m$ matrix $A = (A_{jk})$ and the $r \times s$ matrix $B = (B_{jk})$ as the $nr \times ms$ matrix

$$C = A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{bmatrix} .$$

Denote the information matrix of parameters $\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2$ by $I^{\theta_1\theta_2}$. Note that $I^{\theta_2\theta_1} = \left(I^{\theta_1\theta_2}\right)^T$, and that $(C \otimes D)^T = C^T \otimes D^T$. Then we have from (2.11), (2.12), and (2.13),

$$I^{\alpha_1\alpha_1} = \int_{-\pi}^{\pi} J_{pp}^{T} \otimes f^{T} \otimes Q^{-1} \, d\omega$$

$$I^{\alpha_2\alpha_2} = \int_{-\pi}^{\pi} Q^{-T} \otimes J_{pp} \otimes f \, d\omega$$

$$I^{\alpha_3\alpha_3} = \int_{-\pi}^{\pi} J_{pp} \otimes Q^{-T} \otimes f \, d\omega$$

$$I^{\beta_1\beta_1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} J_{qq} \otimes Q^{-T} \otimes \Sigma \, d\omega$$

$$I^{\beta_2\beta_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma \otimes J_{qq} \otimes Q^{-T} \, d\omega$$

$$I^{\beta_3\beta_3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} J_{qq}^{T} \otimes \Sigma \otimes Q^{-1} \, d\omega$$

$$I^{\alpha_1\beta_3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} J_{qp}^{T} \otimes (\Sigma H^{*} G^{-*})^{T} \otimes Q^{-1} \, d\omega \quad .$$

We have derived the information matrices for three methods of ordering the elements of the parameter matrices. The results for $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\beta}_3$ are used below to find a convenient form for the information matrix of the parameters of a mixed scheme. The results for $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$ are used for autoregressive processes, while those for $\underset{\sim}{\beta}_1$ are used for moving average processes. To summarize, we have

<u>Remarks on Information Matrices</u>

a)
$$I = \begin{bmatrix} I^{\alpha_1\alpha_1} & I^{\alpha_1\beta_3} \\ \text{sym} & I^{\beta_3\beta_3} \end{bmatrix} = \int_{-\pi}^{\pi} \begin{bmatrix} J_{pp}^{T} \otimes f^{T} \otimes Q^{-1} & -\frac{1}{2\pi} J_{qp}^{T} \otimes (\Sigma H^{*} G^{-*})^{T} \otimes Q^{-1} \\ \text{sym} & \frac{1}{2\pi} J_{qq}^{T} \otimes \Sigma \otimes Q^{-1} \end{bmatrix} d\omega$$

$$= \int_{-\pi}^{\pi} \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}^{*} d\omega \quad ,$$

where

$$X = (2\pi)^{-\frac{1}{2}} \underset{\sim}{J}_p \otimes (\Sigma^{\frac{1}{2}} H^* G^{-*})^T \otimes H^{-*} \Sigma^{-\frac{1}{2}}$$

$$Y = -(2\pi)^{-\frac{1}{2}} \underset{\sim}{J}_q \otimes \Sigma^{\frac{1}{2}} \otimes H^{-*} \Sigma^{-\frac{1}{2}} ,$$

$\Sigma^{\frac{1}{2}}$ denotes a positive definite square root of the positive definite matrix Σ , and $\underset{\sim}{J}_r$ is an r-dimensional vector whose kth element is $e^{-ik\omega}$.

b) To find the information matrix of the elements of Σ , and the cross information, in turn, between the $A(\cdot)$ and Σ , and the $B(\cdot)$ and Σ , we note that since Σ is symmetric

$$\frac{\partial f}{\partial \Sigma_{jk}} = \frac{1}{2\pi} C[E_{jk} + (1 - \delta_{jk}) E_{kj}] C^* ,$$

where $C = G^{-1}H$, and $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. Defining

$$D_{jk} = E_{jk} + (1 - \delta_{jk}) E_{kj} ,$$

we have

$$\mathrm{tr}\left[\frac{\partial f}{\partial \Sigma_{jk}} f^{-1} \frac{\partial f}{\partial \Sigma_{\ell m}} f^{-1}\right] = \mathrm{tr}[D_{jk} \Sigma^{-1} D_{\ell m} \Sigma^{-1}] ,$$

which is independent of ω . Thus the element of the information matrix corresponding to $\Sigma_{jk}, \Sigma_{\ell m}$, denoted $I[\Sigma_{jk}, \Sigma_{\ell m}]$, is given by

$$I[\Sigma_{jk}, \Sigma_{\ell m}] = \frac{1}{2}\,\mathrm{tr}[D_{jk}\Sigma^{-1}]$$

$$= \frac{1}{2}\Big[(2 - \delta_{jk} - \delta_{\ell m} + \delta_{jk}\delta_{\ell m})\,\Sigma^{-1}_{j\ell}\,\Sigma^{-1}_{mk} \qquad (2.15)$$

$$+ (2 - \delta_{jk} - \delta_{\ell m})\,\Sigma^{-1}_{k\ell}\,\Sigma^{-1}_{mj}\Big] \quad ,$$

where we consider $\Sigma_{u,v}$, $u \geq v = 1,\ldots,d$ as the distinct elements of Σ. To write this in matrix form, define

$$\underset{\sim}{\sigma}_1 = \mathrm{vec}\,(\Sigma)$$

$$\underset{\sim}{\sigma}_2 = \ell\mathrm{vec}\,(\Sigma) \quad ,$$

where $\ell\mathrm{vec}\,(A)$ is the vec operation on the lower triangular portion of the matrix A. Thus $\underset{\sim}{\sigma}_2$ contains the distinct elements of Σ.

If we treat the d^2 elements of Σ as being distinct (i.e., ignore the symmetry of Σ), then $D_{jk} = E_{jk}$, and it is easy to show that

$$I^{\sigma_1\sigma_1} = \frac{1}{2}\,\Sigma^{-1} \otimes \Sigma^{-1} \quad .$$

Note that we can write

$$\underset{\sim}{\sigma}_1 = A\underset{\sim}{\sigma}_2$$

where A is a $d^2 \times \frac{d(d+1)}{2}$ matrix of zeros and ones. Then (Pagano (1974))

$$
\begin{aligned}
I^{\sigma_2\sigma_2} &= A^T I^{\sigma_1\sigma_1} A \\
&= \frac{1}{2} A^T(\underset{\sim}{\Sigma}^{-1} \otimes \underset{\sim}{\Sigma}^{-1})\, A \quad .
\end{aligned}
\tag{2.16}
$$

Thus the information matrix of $\underset{\sim}{\sigma}_1$ can be found from either (2.15) or (2.16).

c) One can also show that

$$
\begin{aligned}
I[A_{jk}(v), \underset{\sim}{\Sigma}_{\ell m}] &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \operatorname{tr}\left[\frac{\partial f}{\partial A_{jk}(v)}\, f^{-1} \frac{\partial f}{\partial \underset{\sim}{\Sigma}_{\ell m}}\, f^{-1}\right] d\omega \\
&= 0 \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
I[B_{jk}(v), \underset{\sim}{\Sigma}_{\ell m}] &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \operatorname{tr}\left[\frac{\partial f}{\partial B_{jk}(v)}\, f^{-1} \frac{\partial f}{\partial \underset{\sim}{\Sigma}_{\ell m}}\, f^{-1}\right] d\omega \\
&= 0 \quad .
\end{aligned}
$$

These integrals are zero for the same reason that the integrals vanish in the derivation of the information matrices of the $A(\cdot)$ and the $B(\cdot)$.

d) If $q = 0$ we have

$$I^{\alpha_1\alpha_1} = \Gamma_p \otimes \Sigma^{-1} \tag{2.17}$$

$$I^{\alpha_2\alpha_2} = \Sigma^{-1} \otimes \Gamma_p \ , \tag{2.18}$$

since

$$\Gamma_p \equiv \text{BTOEPL}\ (R(0), R(-1), \ldots, R(1-p))$$

$$= \int_{-\pi}^{\pi} J_{pp} \otimes f \, d\omega \ ,$$

where BTOEPL $(R(0),\ldots,R(1-p))$ is a block Toeplitz matrix having $R(j-k)$ in the jth row and kth column of blocks, j , $k = 1,\ldots,p$. Note that (2.18) is the result given by Hannan (1970), p. 329.

e) If $p = 0$, then $Q^{-T} = \frac{1}{2\pi} f^{-T}$ which is of the form of an autoregressive spectral density matrix. Thus $I^{\beta_1\beta_1}$ should be of the same form as $I^{\alpha_1\alpha_1}$ of the previous note with the autocovariances corresponding to the spectral density Q^{-T} replacing $R(\cdot)$ and $2\pi\, \Sigma^{-1}$ replacing Σ . Thus

$$I^{\beta_1\beta_1} = \Gamma i_q \otimes \Sigma \tag{2.19}$$

where

$$\Gamma i_q = \mathrm{BTOEPL}\Big(Ri(0),\ldots,Ri(1-q)\Big) ,$$

$$Ri(v) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} f^{-T}(\omega)\, e^{iv\omega}\, d\omega , \quad v \in Z .$$

See Newton (1975), p. 11, et seq for a discussion of the inverse autocovariances $Ri(\cdot)$. Note that (2.19) is the result stated by Hannan (1970), p. 385.

f) Equations (2.18) and (2.19) also provide an interpretation of information matrices of random matrices (transformed into a random vector by the vec operator) as the covariance matrix of some random vector. We can write

$$I^{\alpha_1\alpha_1} = \{R(j-k) \otimes \Sigma^{-1}\}_{p\mathrm{x}p}$$

and

$$I^{\beta_1\beta_1} = \{Ri(j-k) \otimes \Sigma_i^{-1}\}_{q\mathrm{x}q} ,$$

i.e., $I^{\alpha_1\alpha_1}$ is a $(p \times p)$ block matrix of $(d^2 \times d^2)$ blocks, while $I^{\beta_1\beta_1}$ is a $(q \times q)$ block matrix of $(d^2 \times d^2)$ blocks. Thus there exist d^2-dimensional random vectors $z_1,\ldots,z_p$ and $y_1,\ldots,y_q$ such that $z = \left(z_1^T \vdots \cdots \vdots z_p^T\right)^T$ had covariance matrix given by $I^{\alpha_1\alpha_1}$ and $y = \left(y_1^T \vdots \cdots \vdots y_q^T\right)$ has covariance matrix given by $I^{\beta_1\beta_1}$.

g) Akaike (1973a) derives approximate expressions for the elements of the Hessian of the log likelihood function. From the formulas (2.11),

(2.12), (2.13) above, it is clear that Akaike's formulas for the Hessian are the sample analogues of the information matrix of the mixed scheme parameters. Thus one can use the block Toeplitz matrix inversion techniques developed by Akaike (1973b) to find the asymptotic covariance matrix of the maximum likelihood estimators.

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